Effect of Internal Pressure on an Infinite Cylindrical Shell Subjected to Concentrated Radial Loads

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The effect of internal hydrostatic or lateral pressure on the deformations of an infinitely long, thin, cylindrical isotropic shell subjected to equal concentrated loads equally spaced around a circular cross section is investigated. An exact solution of Flügge's equations for buckling of cylindrical shells, considered herein as equations for small deformations superimposed on large deformations, is obtained. Calculations are shown primarily for the case of two loads, for which the effect of internal pressure is found to be most pronounced.

Nomenclature

- D = flexural stiffness of shell wall, $Eh^3/12(1 - \nu^2)$
- E = Young's modulus of shell material
- h = shell wall thickness
- K = extensional stiffness of shell wall, $Eh/(1 - \nu^2)$
- k = $\frac{1}{12}(h/R)^2$
- N = number of equal and equally spaced radial loads
- n = integer
- P = applied axial load, positive for tension
- p = applied pressure, positive for internal pressure
- Q = magnitude of radial load, positive when inwardly directed
- q_1 = internal pressure parameter, pR^3/D
- q_2 = axial load parameter, $PR/2\pi D$
- R = cylinder middle surface radius
- u,v,w = additional axial, circumferential, and radial displacements, respectively, of point in cylinder middle surface, measured from stressed state
- x = axial distance from radial loads, measured from stressed state
- θ = angular circumferential measurement to point in cylinder middle surface
- ν = Poisson's ratio of shell material
- ξ = x/R
- Φ = displacement function

Introduction

In a recent paper[1] the effects of radial bumper loads on an internally pressurized circular cylindrical shell were approximated by treating the bumper loads as discrete in the circumferential direction but uniformly distributed along generators. In the present paper, the localized nature of the bumper loads in the axial direction as well as in the circumferential direction is taken into account by considering them to be concentrated loads applied to a long, thin, prestressed cylindrical shell.

The equations used in the analysis are those derived by Flügge[2] for buckling of cylindrical shells. These alternatively can be considered to be equations for small deformations superimposed on large deformations. An exact solution is readily obtained for the present problem with the use of a displacement function. Calculations are carried out primarily for the case of two diametrically opposed concentrated loads, for which the effect of internal pressure is found to be most pronounced. Results are given for internal lateral pressure and for internal hydrostatic pressure.

Derivation

A. Differential Equations and Boundary Conditions

We consider an infinitely long, circular, cylindrical, isotropic shell which has been subjected to a large, uniform internal pressure and axial tension and has expanded to a larger circular cylindrical shape. Equal concentrated loads Q equally spaced around a circular cross section are now applied to the shell (see Fig. 1). We assume that the deformations due to the concentrated loads are small compared to those caused by the previously applied internal pressure and axial tension. The loads are assumed to be applied as discontinuities of the transverse shear force. From symmetry of the loading it is apparent that we are concerned with a semi-infinite, circular cylinder to which edge axial forces and Q are applied at discrete points and which is subjected to edge axial forces and bending moments resulting vanishing edge axial displacement and edge rotation (Fig. 2).

Equations appropriate to the problem of small perturbations superimposed upon large deformations of a long circular cylinder are [see Ref. 2, p. 422, Eqs. (7a–7c)]

$$\left\{(1 + kq_2)\frac{\partial^2}{\partial\xi^2} + \left[\frac{1-\nu}{2}(1 + k) + kq_1\right]\frac{\partial^2}{\partial\theta^2}\right\}u + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial\xi\partial\theta} + \left[\nu - k\left(q_1 + \frac{\partial^2}{\partial\xi^2} - \frac{1-\nu}{2}\frac{\partial^2}{\partial\theta^2}\right)\right]\frac{\partial w}{\partial\xi} = 0 \quad (1a)$$

$$\frac{1+\nu}{2}\frac{\partial^2 u}{\partial\xi\partial\theta} + \left\{\left[\frac{1-\nu}{2}(1 + 3k) + kq_2\right]\frac{\partial^2}{\partial\xi^2} + (1 + kq_1)\frac{\partial^2}{\partial\theta^2}\right\}v + \left[1 + k\left(q_1 - \frac{3-\nu}{2}\frac{\partial^2}{\partial\xi^2}\right)\right]\frac{\partial w}{\partial\theta} = 0 \quad (1b)$$

$$\left[\nu - k\left(q_1 + \frac{\partial^2}{\partial\xi^2} - \frac{1-\nu}{2}\frac{\partial^2}{\partial\theta^2}\right)\right]\frac{\partial u}{\partial\xi} + \left[1 + k\left(q_1 - \frac{3-\nu}{2}\frac{\partial^2}{\partial\xi^2}\right)\right]\frac{\partial v}{\partial\theta} + \left\{1 + k\left[\frac{\partial^4}{\partial\xi^4} + 2\frac{\partial^4}{\partial\xi^2\partial\theta^2} + \frac{\partial^4}{\partial\theta^4} + (2 - q_1)\frac{\partial^2}{\partial\theta^2} - q_2\frac{\partial^2}{\partial\xi^2} + 1\right]\right\}w = 0 \quad (1c)$$

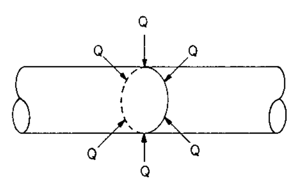

Fig. 1 Concentrated loading applied to infinite cylinder.

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where u, v, and w are, respectively, the small additional longitudinal, circumferential, and radial deformations of the shell middle surface. Let us introduce a displacement function Φ such that

$$u = -\left\{(\nu - kq_1)\left[1 + k\left(3 + \frac{2}{1-\nu}q_2\right)\right]\frac{\partial^2}{\partial\xi^2} - (1 + kq_1)\left(1 + \frac{2}{1-\nu}kq_1\right)\frac{\partial^2}{\partial\xi^2} - k\left[\left(1 + 3k + \frac{2}{1-\nu}q_2k\right)\frac{\partial^4}{\partial\xi^4} - k\left(q_2 - \frac{2}{1-\nu}q_1 + 3\frac{1-\nu}{2}\right)\frac{\partial^4}{\partial\xi^2\partial\theta^2} - (1 + kq_1)\frac{\partial^4}{\partial\theta^4}\right]\right\}\frac{\partial\Phi}{\partial\xi} \quad \text{(2a)}$$

$$v = -\left\{\left[2 + \nu + \frac{3+\nu}{1-\nu}kq_1 + \frac{2}{1-\nu}q_2(1 + kq_1)k\right]\frac{\partial^2}{\partial\xi^2} + (1 + kq_1)\left[1 + k\left(1 + \frac{2}{1-\nu}q_1\right)\right]\frac{\partial^2}{\partial\theta^2} - k\left[\left(2 + \frac{3-\nu}{1-\nu}kq_2\right)\frac{\partial^4}{\partial\xi^4} + \left(2 + \frac{3-\nu}{2}k + \frac{3-\nu}{1-\nu}kq_1\right)\frac{\partial^4}{\partial\xi^2\partial\theta^2}\right]\right\}\frac{\partial\Phi}{\partial\theta} \quad \text{(2b)}$$

$$w = \left\{(1 + kq_2)\left[1 + k\left(3 + \frac{2}{1-\nu}q_2\right)\right]\frac{\partial^4}{\partial\xi^4} + \left[2 + 2(1-\nu)k\left(1 + \frac{3}{4}k\right) + \left(\frac{3-\nu}{1-\nu} + 3k\right)kq_1 + \left(\frac{3-\nu}{1-\nu} + k\right)kq_2 + \frac{4}{1-\nu}k^2q_1q_2\right]\frac{\partial^4}{\partial\xi^2\partial\theta^2} + (1 + kq_1) \times \left[1 + k\left(1 + \frac{2}{1-\nu}q_1\right)\right]\frac{\partial^4}{\partial\theta^4}\right\}\Phi \quad \text{(2c)}$$

Then Eqs. (1a) and (1b) are identically satisfied. Substitution of Eqs. (2) into Eq. (1c) yields an equation for the determination of Φ as

$$\alpha_1\frac{\partial^8\Phi}{\partial\xi^8} + \alpha_2\frac{\partial^8\Phi}{\partial\xi^6\,\partial\theta^2} + \alpha_3\frac{\partial^8\Phi}{\partial\xi^4\partial\theta^4} + \alpha_4\frac{\partial^8\Phi}{\partial\xi^2\partial\theta^6} + \alpha_5\left(\frac{\partial^8\Phi}{\partial\theta^8} + \frac{\partial^6\Phi}{\partial\theta^6}\right) + \beta_1\frac{\partial^6}{\partial\xi^6} + \beta_2\frac{\partial^6\Phi}{\partial\xi^4\partial\theta^2} + \beta_3\frac{\partial^6\theta}{\partial\xi^2\partial\theta^4} + \gamma_1\frac{\partial^4\Phi}{\partial\xi^4} + \gamma_2\frac{\partial^4\Phi}{\partial\xi^2\partial\theta^2} + \gamma_3\left(\frac{\partial^6\Phi}{\partial\theta^6} + \frac{\partial^4\Phi}{\partial\theta^4}\right) = 0 \quad \text{(3)}$$

where

$$\alpha_1 = \left[1 + k\left(3 + \frac{2}{1-\nu}q_2\right)\right][1 - k(1 - q_2)] \quad \text{(4a)}$$

$$\alpha_2 = 4 + k\left(11 - 3\nu + \frac{3-\nu}{1-\nu}q_1 + 2\frac{4-\nu}{1-\nu}q_2\right) + k^2\left[9\frac{1-\nu}{2} + \frac{1-3\nu}{1-\nu}q_1 + \frac{9 - 12\nu - \nu^2}{2(1-\nu)}q_2 + \frac{4}{1-\nu}q_2(q_1 + q_2)\right] \quad \text{(4b)}$$

$$\alpha_3 = 6 + k\left[4 - 3\nu + 3\frac{3-\nu}{1-\nu}(q_1 + q_2)\right] - k^2\left[\nu^2 - \frac{9+\nu}{2}q_2 - \frac{5 - 8\nu - \nu^2}{2(1-\nu)}q_1 + \frac{2}{1-\nu}(q_1^2 + 4q_1q_2 + q_2^2)\right] \quad \text{(4c)}$$

$$\alpha_4 = 4 + k\left[\frac{7-3\nu}{2} + \frac{3-\nu}{1-\nu}(3q_1 + q_2)\right] + k^2\left[3\frac{1-\nu}{2} + \frac{9+\nu}{2}q_1 + q_2 + \frac{4}{1-\nu}q_1(q_1 + q_2)\right] \quad \text{(4d)}$$

$$\alpha_5 = \left[1 + k\left(1 + \frac{2}{1-\nu}q_1\right)\right](1 + q_1k) \quad \text{(4e)}$$

$$\beta_1 = \left[1 + k\left(3 + \frac{2}{1-\nu}q_2\right)\right][2\nu - q_2 - k(2q_1 + q_2^2)] \quad \text{(4f)}$$

$$\beta_2 = 6 - 2q_2 - q_1 + k\left[3(2 - \nu + \nu^2) + 2\frac{5-\nu}{1-\nu}q_2 + \frac{1+3\nu}{1-\nu}q_1 - \frac{3-\nu}{1-\nu}q_2(2q_1 + q_2)\right] + k^2\left[3\frac{3+\nu}{2}q_2 + 3\frac{1-\nu}{2}q_1 + \frac{3+\nu}{1-\nu}q_2^2 + 2\frac{1+\nu}{1-\nu}q_1q_2 - \frac{4}{1-\nu}q_1^2 - \frac{6}{1-\nu}q_1q_2^2\right] \quad \text{(4g)}$$

$$\beta_3 = 2(4-\nu) - 2q_1 - q_2 + k\left[7 - 5\nu + 8\frac{2-\nu}{1-\nu}q_1 + \frac{5-\nu}{1-\nu}q_2 - \frac{3-\nu}{1-\nu}q_1^2 - 2\frac{3-\nu}{1-\nu}q_1q_2\right] + k^2\left[3(1-\nu) + \frac{15+\nu}{2}q_1 + 2q_2 + \frac{5-\nu}{1-\nu}q_1^2 + 2\frac{3+\nu}{1-\nu}q_1q_2 - \frac{6}{1-\nu}q_1^2q_2\right] \quad \text{(4h)}$$

$$\gamma_1 = \left[1 + k\left(3 + \frac{2}{1-\nu}q_2\right)\right]\left[\frac{1-\nu^2}{k} + 1 + q_2 + 2\nu q_1 + k(q_2 - q_1^2)\right] \quad \text{(4i)}$$

$$\gamma_2 = 2(2-\nu) - 3q_1 + q_2 + k\left(7\frac{1-\nu}{2} + 2\frac{3-2\nu}{1-\nu}q_1 + 2\frac{2-\nu}{1-\nu}q_2 - 2\frac{3-\nu}{1-\nu}q_1^2\right) + k^2\left[3\frac{1-\nu}{2} + 3q_1 + q_2 + \frac{4}{1-\nu}q_1\,q_2 - \frac{2}{1-\nu}q_1^2\,(q_1 + q_2)\right] \quad \text{(4j)}$$

$$\gamma_3 = (1 + q_1k)\left[1 + k\left(1 + \frac{2}{1-\nu}q_1\right)\right](1 - q_1) \quad \text{(4k)}$$

The boundary conditions at $\xi = 0$ for our problem are [Ref. 2, p. 233, Eqs. (32)]

$$u = \partial w/\partial x = N_{x\theta} - M_{x\theta}/R = 0 \quad \text{(5a)}$$

$$\overline{Q_x} = Q_x + \frac{\partial M_{x\theta}}{R\partial\theta} = \begin{cases} \lim\limits_{\theta_0\to 0}\dfrac{Q}{4R\theta_0}, & \dfrac{2\pi p}{N} - \theta_0 < \theta < \dfrac{2\pi p}{N} + \theta \\ 0, & \dfrac{2\pi p}{N} + \theta_0 < \theta < \dfrac{2\pi(p+1)}{N} - \theta_0 \end{cases}$$

$$p = 0, 1, 2, \ldots, N - 1 \quad \text{(5b)}$$

where $\overline{Q_x}$ is expressed in Eq. (5b) as the limit of uniform

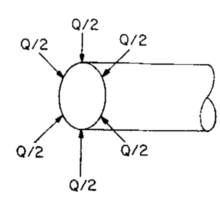

Fig. 2 Semi-infinite cylinder.

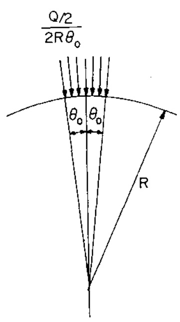

Fig. 3 Concentrated transverse shear loading as the limit of uniformly distributed forces.

transverse shear loads acting over a small portion of the circumference of the middle surface (Fig. 3). A Fourier series expansion of the function is given by

$$\overline{Q_x} = \frac{QN}{2\pi R}\left(\frac{1}{2} + \sum_{n=1}^{\infty} \cos Nn\theta\right) \tag{6}$$

The third of Eqs. (5a) may be reduced considerably if we note that it may be expressed as [Ref. 2, p. 214, Eqs. (9)]

$$\left(N_{x\theta} - \frac{M_{x\theta}}{R}\right)_{\xi=0} = \frac{1-\nu}{2}\frac{K}{R} \times \left[(1+3k)\frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \theta} - 3k\frac{\partial^2 w}{\partial\xi\partial\theta}\right]_{\xi=0} = 0 \tag{7a}$$

But since u and $\partial w/\partial\xi$ vanish at $\xi = 0$, Eq. (7a) may be replaced by

$$(\partial v/\partial\xi)_{\xi=0} = 0 \tag{7b}$$

At infinity, stresses and displacements due to the transverse loads should vanish or be bounded.

With the use of Eqs. (3), the edge conditions at $\xi = 0$ given by Eqs. (6a) and (7b) may be written as

$$\left[(1+kq_1)\left(1 + \frac{2}{1-\nu}q_1 k\right)\frac{\partial^2}{\partial\theta^2} - k(1+kq_1)\frac{\partial^4}{\partial\theta^4}\right]\frac{\partial\Phi}{\partial\xi} - \left\{(\nu - kq_1)\left[1 + k\left(3 + \frac{2}{1-\nu}q_2\right)\right] + k^2\left(q_2 - \frac{2}{1-\nu}q_1 + 3\frac{1-\nu}{2}\right)\frac{\partial^2}{\partial\theta^2}\right\}\frac{\partial^3\Phi}{\partial\xi^3} + k\left[1 + k\left(3 + \frac{2}{1-\nu}q_2\right)\right]\frac{\partial^5\Phi}{\partial\xi^5} = 0 \tag{8a}$$

$$\left\{(1+kq_1)\left[1 + k\left(1 + \frac{2}{1-\nu}q_1\right)\right]\frac{\partial^4}{\partial\theta^4}\right\}\frac{\partial\Phi}{\partial\xi} + 2\left\{1 + k\left[1 - \nu + \frac{1}{2}\frac{3-\nu}{1-\nu}(q_1+q_2)\right] + k^2\left[3\frac{1-\nu}{4} + \frac{1}{2}(3q_1+q_2) + \frac{2}{1-\nu}q_1q_2\right]\frac{\partial^2}{\partial\theta^2}\right\}\frac{\partial^3\Phi}{\partial\xi^3} + (1+kq_2)\left[1 + k\left(3 + \frac{2}{1-\nu}q_2\right)\right]\frac{\partial^5\Phi}{\partial\xi^5} = 0 \tag{8b}$$

$$\left\{(1+kq_1)\left[1 + k\left(1 + \frac{2}{1-\nu}q_1\right)\right]\frac{\partial^3}{\partial\theta^3}\right\}\frac{\partial\Phi}{\partial\xi} + \left\{\left[2 + \nu + k\left(\frac{3+\nu}{1-\nu}q_1 + \frac{2}{1-\nu}q_2\right) + \frac{2}{1-\nu}k^2q_1q_2\right]\frac{\partial}{\partial\theta} - k\left[2 + \frac{3-\nu}{2}k\left(1 + \frac{2}{1-\nu}q_1\right)\right]\frac{\partial^3}{\partial\theta^3}\right\}\frac{\partial^3\Phi}{\partial\xi^3} - \left[k\left(2 + \frac{3-\nu}{1-\nu}kq_2\right)\frac{\partial}{\partial\theta}\right]\frac{\partial^5\Phi}{\partial\xi^5} = 0 \tag{8c}$$

The solution of Eq. (4) which yields stresses and deformations which are periodic with period $2\pi/N$ around the circumference can be expressed as

$$\Phi = \tilde{\Phi}(\theta,\xi) + \sum_{n=\theta}^{\infty} \Phi_{Nn}(\xi)\cos Nn\theta \tag{9a}$$

where

$$\tilde{\Phi}(\theta,\xi) = \frac{\alpha_0 + 6(\nu - kq_1)(1 + \{3 + [2/(1-\nu)]q_2\}k)\beta_0}{2(1+kq_1)\{1 + [2/(1-\nu)]kq_1\}} \times \theta^2\xi + \beta_0\xi^3 \tag{9b}$$

and α_0 and β_0 are arbitrary constants of integration. Then Eqs. (8) imply the following edge conditions:

$$\frac{d\Phi_{Nn}(0)}{d\xi} = \frac{d^3\Phi_{Nn}(0)}{d\xi^3} = \frac{d^5\Phi_{Nn}(0)}{d\xi^5} = 0 \quad (n = 1,2,3,\ldots) \tag{10a}$$

$$\alpha_0 - (\nu - kq_1)\left[1 + \left(3 + \frac{2}{1-\nu}q_2\right)k\right]\frac{d^3\Phi_0(0)}{d\xi^3} = \frac{d^5\Phi_0(0)}{d\xi^5} = 0 \tag{10b}$$

Equations (6c) and (10), together with the relation [Ref. 2, p. 209, Eq. (1e) and p. 214, Eqs. (9a–9h)]

$$\overline{Q_x} = -\frac{D}{R^3}\left[\frac{\partial^3 w}{\partial\xi^3} + (2-\nu)\frac{\partial^3 w}{\partial\xi\partial\theta^2} - \frac{\partial^2 u}{\partial\xi^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial\theta^2} - \frac{3-\nu}{2}\frac{\partial^2 v}{\partial\xi\partial\theta}\right] \tag{11}$$

imply the remaining edge conditions

$$d^7\Phi_{Nn}(0)/d\xi^7 = -\delta \quad (n = 1,2,3,\ldots) \tag{12a}$$

$$d^7\Phi_0(0)/d\xi^7 = -\tfrac{1}{2}\delta \tag{12b}$$

where

$$\delta = \frac{NQR^2}{2\pi D}\frac{1}{\{1 + k[3 + 2/(1-\nu)q_2]\}[1 - k(1-q_2)]} \tag{12c}$$

B. Solution of the Differential Equation

Let us consider the solutions of Eq. (4) of the form

$$\Phi = e^{\lambda_{Nn}\xi}\cos Nn\theta \tag{13a}$$

Then λ_{Nn} is determined by the equation

$$\alpha_1\lambda_{Nn}^8 + [\beta_1 - \alpha_2(Nn)^2]\lambda_{Nn}^6 + [\gamma_1 - \beta_2(Nn)^2 + \alpha_3(Nn)^4]\lambda_{Nn}^4 - (Nn)^2[\gamma_2 - (Nn)^2\beta_3 + (Nn)^4\alpha_4]\lambda_{Nn}^2 + (Nn)^4[(Nn)^2 - 1][(Nn)^2\alpha_5 - \gamma_3] = 0 \tag{13b}$$

For each value of $n (n \neq 0)$ there are eight roots of Eq. (13b). For the case of internal pressure and axial tension ($q_1 > 0$, $q_2 > 0$), calculations indicate that the roots are of the following forms.†

Case I (eight complex roots):

$$\lambda_{Nnr} = \pm p_{Nn1} \pm iq_{Nn1} \quad (r = 1,2,3,4) \tag{14a}$$

$$\lambda_{Nnr} = \pm p_{Nn2} \pm iq_{Nn2} \quad (r = 5,6,7,8) \tag{14b}$$

Case II (two pair of equal real roots, four complex roots):

$$\lambda_{Nnr} = \pm p_{Nn1} \quad (r = 1,2,3,4) \tag{15a}$$

$$\lambda_{Nnr} = \pm p_{Nn2} \pm iq_{Nn2} \quad (r = 5,6,7,8) \tag{15b}$$

Case III (two pair of unequal real roots, four complex roots):

$$\lambda_{Nnr} = \pm p_{Nn1} \quad (r = 1,2) \tag{16a}$$

$$\lambda_{Nnr} = \pm q_{Nn1} \quad (r = 3,4) \tag{16b}$$

† These forms were found to be necessary for the case $q_1 = q_2 = 0$, contrary to the common supposition that all roots are complex for the unpressurized cylinder.

$$\lambda_{Nnr} = \pm p_{Nn2} \pm iq_{Nn2} \ (r = 5,6,7,8) \quad (16c)$$

For each case the function $\Phi_{Nn}(\xi)$ has a different form. We write these as

Case I:

$$\Phi_{Nn}(\xi) = \delta \sum_{j=1}^{2} (A_{Nnj} \cos q_{Nnj}\xi + B_{Nnj} \sin q_{Nnj}\xi) e^{-p_{Nnj}\xi} \quad (17a)$$

Case II:

$$\Phi_{Nn}(\xi) = \delta[(A_{Nn1} + B_{Nn1}\xi)e^{-p_{Nn1}\xi} + (A_{Nn2} \cos q_{Nn2}\xi + B_{Nn2} \sin q_{Nn2}\xi)e^{-p_{Nn2}\xi}] \quad (17b)$$

Case III:

$$\Phi_{Nn}(\xi) = \delta[A_{Nn1}e^{-p_{Nn1}\xi} + B_{Nn1}e^{-q_{Nn1}\xi} + (A_{Nn2} \cos q_{Nn2}\xi + B_{Nn2} \sin q_{Nn2}\xi)e^{-p_{Nn2}\xi}] \quad (17c)$$

In each case the deformations and stresses decay exponentially with increasing longitudinal distance from the loads.

The coefficients A_{Nn1}, B_{Nn1}, A_{Nn2}, B_{Nn2} may be determined through the use of Eqs. (10a) and (12a). After some manipulation we can obtain the following.

Case I:

$$A_{Nn1} = (1/p_{Nn1})A(p_{Nn1},q_{Nn1},p_{Nn2},q_{Nn2}) \quad (18a)$$

$$B_{Nn1} = (1/q_{Nn1})A(q_{Nn1},p_{Nn1},q_{Nn2},p_{Nn2}) \quad (18b)$$

$$A_{Nn2} = (1/p_{Nn2})A(p_{Nn2},q_{Nn2},p_{Nn1},q_{Nn1}) \quad (18c)$$

$$B_{Nn2} = (1/q_{Nn2})A(q_{Nn2},p_{Nn2},q_{Nn1},p_{Nn1}) \quad (18d)$$

with

$$A(x_1,x_2,x_3,x_4) = -\frac{5x_1^4 - 10x_1^2x_2^2 + x_2^4 - 2(3x_1^2 - x_2^2)(x_3^2 - x_4^2) + (x_3^2 + x_4^2)^2}{2(x_1^2 + x_2^2)[(x_1^2 - x_2^2 - x_3^2 + x_4^2)^2 + 4(x_1x_2 + x_3x_4)^2]\,[(x_1^2 - x_2^2 - x_3^2 + x_4^2)^2 + 4(x_1x_2 - x_3x_4)^2]} \quad (18e)$$

Case II:

$$A_{Nn1} = (1/p_{Nn1})A(p_{Nn1},0,p_{Nn2},q_{Nn2}) \quad (19a)$$

$$B_{Nn1} = A(0,p_{Nn1},q_{Nn2},p_{Nn2}) \quad (19b)$$

$$A_{Nn2} = (1/p_{Nn2})A(p_{Nn2},q_{Nn2},p_{Nn1},0) \quad (19c)$$

$$B_{Nn2} = (1/q_{Nn2})A(q_{Nn2},p_{Nn2},0,p_{Nn1}) \quad (19d)$$

Case III:

$$A_{Nn1} = \frac{q_{Nn1}^4 - 2q_{Nn1}^2(p_{Nn2}^2 - q_{Nn2}^2) + (p_{Nn2}^2 + q_{Nn2}^2)^2}{p_{Nn1}(p_{Nn1}^2 - q_{Nn1}^2)\Delta_{Nn}} \quad (20a)$$

$$B_{Nn1} = \frac{p_{Nn1}^4 - 2p_{Nn1}^2(p_{Nn2}^2 - q_{Nn2}^2) + (p_{Nn2}^2 + q_{Nn2}^2)^2}{q_{Nn1}(q_{Nn1}^2 - p_{Nn1}^2)\Delta_{Nn}} \quad (20b)$$

$$A_{Nn2} = -\frac{5p_{Nn2}^4 - 10p_{Nn2}^2q_{Nn2}^2 + q_{Nn2}^4 - (3p_{Nn2}^2 - q_{Nn2}^2)(p_{Nn1}^2 + q_{Nn1}^2) + p_{Nn1}^2q_{Nn1}^2}{2p_{Nn2}(p_{Nn2}^2 + q_{Nn2}^2)\Delta_{Nn}} \quad (20c)$$

$$B_{Nn2} = -\frac{5q_{Nn2}^4 - 10p_{Nn2}^2q_{Nn2}^2 + p_{Nn2}^4 - (p_{Nn2} - 3q_{Nn2}^2)(p_{Nn1}^2 + q_{Nn1}^2) + p_{Nn1}^2q_{Nn1}^2}{2q_{Nn2}(p_{Nn2}^2 + q_{Nn2}^2)\Delta_{Nn}} \quad (20d)$$

$$\Delta_{Nn} = [(p_{Nn2}^2 + q_{Nn2}^2)^2 - (p_{Nn2}^2 - q_{Nn2}^2) \times (p_{Nn1}^2 + q_{Nn1}^2) + p_{Nn1}^2q_{Nn1}^2]^2 + 4p_{Nn2}^2q_{Nn2}^2(p_{Nn1}^2 - q_{Nn1}^2)^2 \quad (20e)$$

When n is equal to zero, four of the roots of Eq. (13b) vanish. The remaining roots belong to one of the following cases:

Case Ia (4 complex roots):

$$\lambda_{Nnr} = \pm p_{01} \pm iq_{01} \ (r = 1,2,3,4) \quad (21)$$

Case IIa (two pairs of equal real roots):

$$\lambda_{Nnr} = \pm p_{01} \ (r = 1,2) \quad (22a)$$

$$\lambda_{Nnr} = \pm p_{01} \ (r = 3,4) \quad (22b)$$

Case IIIa (two pairs of unequal real roots):

$$\lambda_{Nnr} = \pm p_{01} \ (r = 1,2) \quad (23a)$$

$$\lambda_{Nnr} = \pm q_{01} \ (r = 3,4) \quad (23b)$$

The solutions may then be written for each case as

Case Ia:

$$\Phi_0(\xi) = \delta e^{-p_{01}\xi} (A_{01} \cos q_{01}\xi + B_{01} \sin q_{01}\xi) \quad (24a)$$

with

$$A_{01} = -\frac{5p_{01}^4 - 10p_{01}^2q_{01}^2 + q_{01}^4}{4p_{01}(p_{01}^2 + q_{01}^2)^5} \quad (24b)$$

$$B_{01} = \frac{p_{01}^4 - 10p_{01}^2q_{01}^2 + 5q_{01}^4}{4q_{01}(p_{01}^2 + q_{01}^2)^5} \quad (24c)$$

Case IIa:

$$\Phi_0(\xi) = \delta(A_{01} + B_{01}\xi)e^{-p_{01}\xi} \quad (25a)$$

with

$$A_{01} = -5/4p_{01}^7 \quad (25b)$$

$$B_{01} = -1/4p_{01}^6 \quad (25c)$$

Case IIIa:

$$\Phi_0(\xi) = \delta(A_{01}e^{-p_{01}\xi} + B_{01}e^{-q_{01}\xi}) \quad (26a)$$

with

$$A_{01} = 1/2p_{01}^5(p_{01}^2 - q_{01}^2) \quad (26b)$$

$$B_{01} = 1/2q_{01}^5(q_{01}^2 - p_{01}^2) \quad (26c)$$

From Eqs. (3, 9, and 17–26), we may find the displacements of the shell middle surface. The radial deformation of the cylinder, for example, is given by

$$\frac{2\pi Dw}{NQR^2} = \sum_{n=0}^{\infty} \psi_{Nn}(\xi) \cos Nn\theta \quad (27)$$

where $\psi_{Nn}(\xi)$ has the following forms for the various cases.

Case I:

$$\psi_{Nn}(\xi) = \sum_{j=1}^{2} \psi_{Nnj}(\xi) \quad (28a)$$

where

$$\psi_{Nnj}(\xi) = \{[K_1(p_{Nnj}^4 - 6p_{Nnj}^2q_{Nnj}^2 + q_{Nnj}^4) - 2(Nn)^2K_2(p_{Nnj}^2 - q_{Nnj}^2) + K_3(Nn)^4]\chi_{Nnj}^{(1)}(\xi) + 4p_{Nnj}q_{Nnj}[K_1(p_{Nnj}^2 - q_{Nnj}^2) - K_2(Nn)^2]\chi_{Nnj}^{(2)}(\xi)\} \quad (28b)$$

$$\chi_{Nnj}^{(1)}(\xi) = (A_{Nnj} \cos q_{Nnj} + B_{Nnj} \sin q_{Nnj}\xi)e^{-p_{Nnj}\xi} \quad (28c)$$

$$\chi_{Nnj}^{(2)}(\xi) = (A_{Nnj} \sin q_{Nnj}\xi - B_{Nnj} \cos q_{Nnj}\xi)e^{-p_{Nnj}\xi} \quad (28d)$$

$$K_1 = (1 + q_2k)/[1 - k(1 - q_2)] \quad (28e)$$

$$K_2 = \frac{1 + k\left[1 - \nu + \dfrac{3-\nu}{2(1-\nu)}(q_1 + q_2)\right] + k^2\left(3\dfrac{1-\nu}{4} + \dfrac{3}{2}q_1 + \dfrac{1}{2}q_2 + \dfrac{2}{1-\nu}q_1q_2\right)}{[1 - k(1-q_2)]\left[1 + K\left(3 + \dfrac{2}{1-\nu}q_2\right)\right]} \tag{28f}$$

$$K_3 = \frac{(1 + q_1k)(1 + k\{1 + [2/(1-\nu)]q_1\})}{[1 - k(1-q_2)](1 + k\{3 + [2/(1-\nu)]q_2\})} \tag{28g}$$

Case II:

$$\psi_{Nn}(\xi) = [K_1p_{Nn1}{}^4 - 2(Nn)^2K_2p_{Nn1}{}^2 + k_3(Nn)^4](A_{Nn1} + B_{Nn1}\xi)e^{-p_{Nn1}\xi} - 4p_{Nn1}[K_1p_{Nn1}{}^2 - K_2(Nn)^2]B_{Nn1}e^{-p_{Nn1}\xi} + \psi_{Nn2}(\xi) \tag{29}$$

Case III:

$$\psi_{Nn}(\xi) = [K_1p_{Nn1}{}^4 - 2(Nn)^2K_2p_{Nn1}{}^2 + K_3(Nn)^4] \times A_{Nn1}e^{-q_N\ \xi} + [K_1q_{Nn1}{}^4 - 2(Nn)^2K_2q_{Nn1}{}^2 + K_3(Nn)^4] \times B_{Nn1}e^{-q_{Nn1}\xi} + \psi_{Nn2}(\xi) \tag{30}$$

Case Ia:

$$\psi_0(\xi) = K_1[(p_{01}{}^4 - 6p_{01}{}^2q_{01}{}^2 + q_{01}{}^4)\chi_{01}{}^{(1)}(\xi) + 4p_{01}q_{01}(p_{01}{}^2 - q_{01}{}^2)\chi_{01}{}^{(2)}(\xi)] \tag{31}$$

Case IIa:

$$\psi_0(\xi) = K_1[p_{01}{}^4(A_{01} + B_{01}\xi)e^{-p_{01}\xi} - 4p_{01}{}^3B_{01}e^{-p_{01}\xi}] \tag{32}$$

Case IIIa:

$$\psi_0(\xi) = K_1[p_{01}{}^4A_{01}e^{-p_{01}\xi} + q_{01}{}^4B_{01}e^{-q_{01}\xi}] \tag{33}$$

Stress resultants may be found by means of Eqs. (9a–h) of Ref. 2, p. 214.

The solution is very easily modified to include the case of distributed radial ring loading with period $2\pi/N$ by replacing δ in Eqs. (17, 24a, and 25a) by the coefficient of $\cos n N\theta$ in the Fourier series for the distributed loading (see Ref. 3, for example).

Results and Discussion

Calculations were carried out to determine the effect of internal pressure on the radial displacements of a cylinder subjected to diametrically opposed concentrated loads. The values of the parameters used were

$$\nu = 0.3, \quad N = 2, \quad R/h = 10, 100, 1000$$

Both internal hydrostatic pressure ($q_2 = \frac{1}{2}q_1$) and internal lateral pressure ($q_2 = 0$) were considered. The procedure used for a particular shell subjected to a particular pressure was to solve the characteristic equation (13b) for each value of n considered. The roots were then substituted in Eqs. (18, 19, or 20 and 24, 25, or 26) to determine the coefficients A_{n1}, A_{n2}, B_{n1}, B_{n2}, A_{01}, and B_{01}. Finally $\psi_{Nn}(\xi)$ and $\psi_0(\xi)$ were calculated from Eqs. (28, 29, or 30) and (31, 32, or 33)

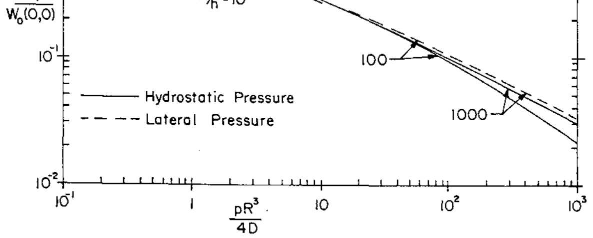

Fig. 4 Variation of radial deflection under load with internal pressure ($N = 2$, $\nu = 0.3$).

for ξ equal to $3\beta(R/h)^{1/2}$. The parameter β was varied by steps of 0.01 from 0 to 0.1 and by steps of 0.1 from 0.1 to 1. The number of terms used (the maximum value of n) was generally 20 for R/h equal to 10 and 100, and 30 for R/h equal to 1000.

Convergence of the infinite series for the radial displacements becomes poorer as the internal pressure increases and as the shell becomes thinner. In cases when convergence was not achieved with 20 or 30 terms, the sum of the remaining terms was estimated by means of a geometric series approximation since the ratio of the consecutive terms was almost constant. The results are shown in Table 1.

For no internal pressure the analysis coincides with that of Ref. 4, for example. The variation of the deflection parameter $Dw_0(0,0)/QR^2$ is represented closely by the relationship

$$\pi Dw_0(0,0)/QR^2 = 0.215(R/h)^{-1/2} \tag{34a}$$

or by

$$Ehw_0(0,0)/Q = 0.746(R/h)^{3/2} \tag{34b}$$

The effect of internal pressure on the radial deflection under

Table 1 Variation of radial deflection under load with internal pressure ($N = 2$, $\nu = 0.3$)

		Internal hydrostatic pressure		Internal lateral pressure	
R/h	$\frac{pR^3}{4D}$	$\frac{\pi Dw(0,0)}{QR^2}$	$\frac{w(0,0)}{w_0(0,0)}$	$\frac{\pi Dw(0,0)}{QR^2}$	$\frac{w(0,0)}{w_0(0,0)}$
	0.0	0.06647	1.000	0.06648	1.000
	0.025	0.06532	0.983	0.06535	0.983
	0.05	0.06420	0.966	0.06428	0.967
	0.075	0.06316	0.950	0.06327	0.952
	0.1	0.06216	0.935	0.06230	0.937
	0.2	0.05857	0.881	0.05883	0.885
	0.3	0.05558	0.836	0.05592	0.841
	0.4	0.05300	0.797	0.05344	0.804
10	0.5	0.05076	0.764	0.05127	0.771
	0.6	0.04880	0.734	0.04935	0.742
	0.7	0.04706	0.708	0.04766	0.717
	0.8	0.04547	0.684	0.04615	0.694
	0.9	0.04406	0.663	0.04477	0.673
	1.0	0.04277	0.643	0.04352	0.655
	2.0	0.03416	0.514	0.03517	0.529
	10.0	0.01712	0.258	0.01899	0.286
	0.	0.02145	1.000	0.02145	1.000
	1.	0.01386	0.646	0.01391	0.648
	2.5	0.01031	0.480	0.01033	0.482
	5.	0.00791	0.368	0.00792	0.369
	7.5	0.00670	0.312	0.00675	0.315
	10	0.00593	0.276	0.00599	0.279
	20	0.00435	0.203	0.00444	0.207
	30	0.00361	0.168	0.00372	0.173
100	40	0.00313	0.146	0.00324	0.151
	50	0.00283	0.132	0.00294	0.137
	60	0.00257	0.120	0.00271	0.126
	70	0.00237	0.110	0.00251	0.117
	80	0.00222	0.103	0.00235	0.110
	90	0.00208	0.0970	0.00224	0.104
	100	0.00197	0.0918	0.00214	0.100
	200	0.00134	0.0625	0.00153	0.0714
	1000	0.000479	0.0223	0.000736	0.0343
	0	0.006785	1.000	0.006785	1.000
	10	0.001880	0.277	0.001882	0.277
1000	50	0.000915	0.135	0.000918	0.135
	100	0.000656	0.0964	0.000658	0.0965
	1000	0.000213	0.0314	0.000229	0.0337

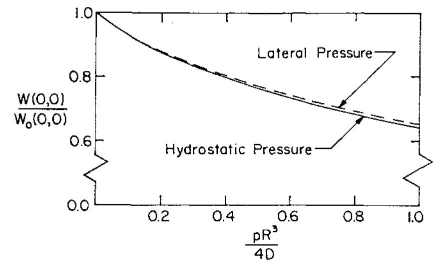

Fig. 5 Expanded scale for effect of internal pressure on radial displacement under load ($N = 2$, $\nu = 0.3$).

the load is shown in Fig. 4. We find that the parameter $pR^3/4D$ is a reasonably good correlation parameter for the ratio of the deflection of the pressurized shell and of the unpressurized shell $w(0,0)/w_0(0,0)$ when the internal pressure is less than that which causes yielding of the shell material $[pR^3/4D \lesssim (R/10h)^2$ or $pR/Eh \lesssim 0.00366]$. In this range of pressure there is little difference between the effects of internal hydrostatic pressure or internal lateral pressure. The variation of radial deflection in the pressure range $0 < pR^3/4D < 1$ is shown in Fig. 5. The maximum reduction in the radial deflection in this range is about 35%.

The effect of internal pressure on the longitudinal variation of the radial deflection along the generator passing through a

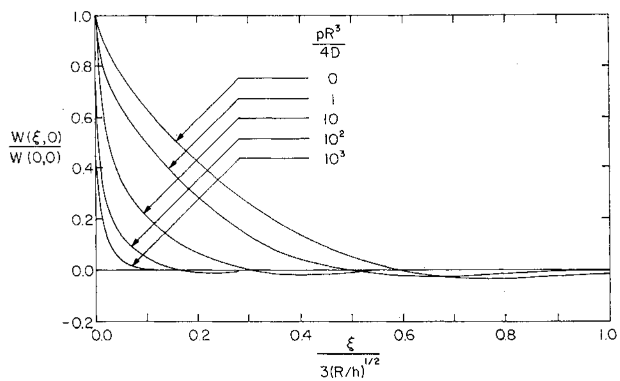

Fig. 6 Variation of radial deformation curves for loaded generator with internal pressure ($N = 2$, $\nu = 0.3$, $R/h = 100$).

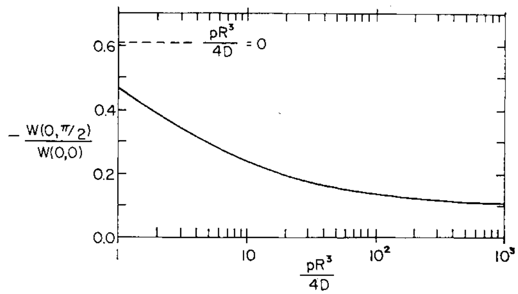

Fig. 7 Effect of internal pressure on radial deformation midway between loads ($N = 2$, $\nu = 0.3$, $R/h = 100$).

load is shown in Fig. 6. The curves vary so rapidly in the vicinity of the concentrated load that the horizontal tangent under the load cannot be plotted for the chosen scale. We see that increasing internal pressure causes the deformation pattern to become more localized. Although the curves shown are for $R/h = 100$, and for internal hydrostatic pressure, curves for other values of R/h and for internal lateral pressure are almost identical. The effect of internal pressure is to cause the deformation pattern to become somewhat more localized in the circumferential direction as well. This effect is indicated by the plot of the ratio of the radial deformation at the point $\xi = 0$, $\theta = \pi/2$ and the radial deflection under the load shown in Fig. 7. We see that the radial deflection midway between the loads decreases from a value about 0.6 of the maximum deflection to a value about 0.1 as the pressure parameter increases from 0 to 1000.

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